Boundary layers whose streamlines are closed

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SUMMARY

Certain two-dimensional, laminar boundary layers are considered whose streamlines are closed. The speed at the solid boundary is supposed uniform, the boundary outline being stationary, and the speed in the boundary layer is supposed to differ only slightly from that of the boundary. A formal solution is then obtained for the motion in the boundary layer. The analysis confirms that a closed boundary layer may exist and yields a condition needed to determine the inviscid motion. The condition is extracted in a simplified but approximate form and two examples of its use are given. A further class of closed boundary layers, namely those for which the pressure is uniform, is also considered. For this class the condition needed to determine the inviscid motion may be derived in a form both simple and exact.

1. INTRODUCTION

In the steady motion of a slightly viscous incompressible fluid the direct effect of local viscous forces may generally be neglected except in the neighbourhood of certain singular surfaces. The indirect effect of the viscous forces that act in these layers is, however, often appreciable throughout the whole region of motion. In such cases, the motion of the fluid in the 'inviscid' region cannot be determined independently of the fluid in the viscous shear layers.

In two-dimensional motions, indeterminacies which arise from the complete neglect of viscous action are usually of one of two kinds. Either a shear layer is shed from a bounding surface and the inviscid motion is indeterminate because the position of this shear layer is not known. Or a shear layer, whose position is known or can be regarded as known, is closed and the inviscid motion is indeterminate to the extent of certain constants. It is with this second kind of indeterminacy that we shall be concerned.

For irrotational motions, such indeterminacies generally take the form that the circulation round each closed boundary is unknown. For instance, if fluid of infinite extent streams past a circular cylinder supposedly rotating so rapidly that the boundary layer does not separate, then the inviscid motion is undertermined to the extent of its circulation about the cylinder.

For rotational motions, the vorticity may also be undetermined. As a typical example may be cited the motion between two rotating circular cylinders whose axes are set apart. Here the direct action of viscosity may be supposed confined to boundary layers at the cylinders. Since the streamlines of the inviscid motion are closed, the vorticity of the inviscid motion is uniform (Batchelor 1956a). The value of this vorticity and the value of the (inviscid) circulation round some given circuit enclosing the inner cylinder are then undetermined. Flow patterns similar to this have recently been suggested as representing the motion in closed regions divided from a main flow by a separated boundary layer (Squire 1956, Batchelor 1956b).

To resolve the indeterminacies in steady motions of this kind, it is necessary to examine shear layers whose streamlines are closed. Only one solution (Squire 1956) for the velocity distribution in a closed shear layer has so far been reported. In this case the motion in the shear layer was calculated using a Rayleigh-type approximation, the convective velocity in the momentum equation being replaced by the tangential velocity at its outer edge.

In the following a class of two-dimensional boundary layers is considered for which the motion in the boundary layer can, in principle, be solved exactly. The closed solid boundary is supposed to move tangentially with uniform speed, and the speed in the boundary layer is assumed to differ only slightly from that of the boundary. A formal solution may then be derived by expanding the velocity in a power series in a small parameter representative of the small differences of speed through the boundary layer. As may be expected, the analysis of the closed boundary layer enables the associated indeterminacy of the inviscid motion to be resolved, in principle, exactly.

Those closed boundary layers in which the pressure is uniform, the boundary speed being non-uniform, are also considered. For these boundary layers the information needed to determine the inviscid motion can be extracted without solving in detail for the motion in the boundary layer.

2. Series solution

We start by deriving a series solution for closed boundary layers of the first class in which the speed is almost uniform.

The motion in the boundary layer is assumed steady and two-dimensional, the fluid being incompressible and of kinematic viscosity ν. Each streamline is assumed to circumscribe the closed boundary and, to the boundary layer approximation, to have the common length $2\pi L$. The velocity at the boundary is supposed tangential and of uniform magnitude u_0. For definiteness, the closed boundary may be visualized as an inextensible band which lines a fixed cylinder and moves transversely to its generators. Let xL be arc-distance along the boundary from some fixed reference point. Further let $\sqrt{(\nu u_0 L)}\psi$ and $u_0 u(x,\psi)$ be the stream function and tangential component of velocity in the boundary layer. With x, ψ as coordinates, the equations that govern the motion in the boundary layer may then be written

$$\frac{\partial u^2}{\partial x} = \frac{dU^2}{dx} + u\frac{\partial^2 u^2}{\partial \psi^2}, \tag{1}$$

the boundary conditions at the moving boundary $\psi = 0$ (say) and at the outer edge of the boundary layer being, respectively,

$$u(x, 0) = 1, \qquad \lim_{\psi\to\infty} u(x, \psi) = U(x). \tag{2}$$

Since the velocity on each closed streamline returns cyclicly to its initial value, we have the additional condition that, for each closed streamline,

$$u(0, \psi) = u(2\pi, \psi). \tag{3}$$

The assumptions that underlie the series expansions of the velocity in the boundary layer are best introduced by referring to the whole motion of which the boundary layer is part. A parameter γ is presumed to be formed from the lengths and velocities that enter into the data for the complete motion. For a certain value of γ ($\gamma = 0$, say), the motion is supposed such that there is no shear layer at the boundary considered, the velocity of the inviscid motion exactly matching that of the boundary. The motion is then considered for the small, non-zero values of γ for which the velocity of the inviscid motion at the boundary is slightly different from that of the boundary. For these values of γ it is supposed that U^2 may be represented in the form

$$U^2(x) = 1 + \sum_{n=1}^{\infty} Q_n(x)\gamma^n. \tag{4}$$

A similar expansion for u^2 may then be expected to hold in the boundary layer. Accordingly we write

$$u^2(x, \psi) = 1 + \sum_{n=1}^{\infty} q_n(x, \psi)\gamma^n. \tag{5}$$

Since the boundary layer equation involves u linearly we also write

$$u(x, \psi) = \left\{1 + \sum_{n=1}^{\infty} q_n(x, \psi)\gamma^n\right\}^{1/2} = 1 + \sum_{n=1}^{\infty} P_n\gamma^n, \tag{6}$$

where the functions P_n are polynomials in $q_1, q_2, \ldots, q_n$. The viscous term of the boundary layer equation is then

$$u\frac{\partial^2 u^2}{\partial \psi^2} = \frac{\partial^2 q_1}{\partial \psi^2} + \sum_{n=2}^{\infty}\left(\frac{\partial^2 q_n}{\partial \psi^2} + R_n\right)\gamma^n, \tag{7}$$

where

$$R_n = \sum_{r=1}^{n-1} P_{n-r}\frac{\partial^2 q_r}{\partial \psi^2}. \tag{8}$$

The remainder terms R_n depend only on the lower order coefficients $q_1, q_2, \ldots, q_{n-1}$ of the expansion for u^2 and may readily be calculated from (6) and (8). In particular,

$$\left.\begin{aligned} R_2 &= \tfrac{1}{2}q_1\frac{\partial^2 q_1}{\partial \psi^2}, \\ R_3 &= \tfrac{1}{2}q_1\frac{\partial^2 q_2}{\partial \psi^2} + (\tfrac{1}{2}q_2 - \tfrac{1}{8}q_1^2)\frac{\partial^2 q_1}{\partial \psi^2}. \end{aligned}\right\} \tag{9}$$

The boundary layer equation (1) and boundary conditions (2) may now be replaced by a set of equations and boundary conditions for the coefficients q_n. Thus we have

$$\left.\begin{aligned} \frac{\partial q_1}{\partial x} &= \frac{dQ_1}{dx} + \frac{\partial^2 q_1}{\partial \psi^2}, \\ \frac{\partial q_n}{\partial x} &= \frac{dQ_n}{dx} + \frac{\partial^2 q_n}{\partial \psi^2} + R_n(q_1, q_2, \ldots, q_{n-1}) \qquad (n > 1), \end{aligned}\right\} \tag{10}$$

with the boundary conditions, for all n,

$$q_n(x, 0) = 0, \qquad \lim_{\psi\to\infty} q_n(x, \psi) = Q_n(x), \qquad q_n(0, \psi) = q_n(2\pi, \psi). \tag{11}$$

Since the velocity gradient $u\,\partial u/\partial\psi$ must vanish at the outer edge of the boundary layer, we also have that

$$\lim_{\psi\to\infty} \frac{\partial q_n}{\partial\psi}(x, \psi) = 0.$$

At first sight, it appears that these equations enable all the coefficients q_n to be calculated in turn, thereby determining the motion in the boundary layer. In fact, however, unless the distribution of the external velocity round the boundary layer is suitably restricted, none of these equations possesses a solution which satisfies its boundary conditions. For, on integrating (10) round a closed streamline, we find that

$$\frac{\partial^2}{\partial\psi^2}\int_0^{2\pi} q_1\,dx = 0,$$
$$\frac{\partial^2}{\partial\psi^2}\int_0^{2\pi} q_n\,dx = -\int_0^{2\pi} R_n\,dx \qquad (n > 1),$$

whence, on integrating twice with respect to ψ and using the boundary conditions (11) for q_n,

$$\int_0^{2\pi} Q_1\,dx = 0, \tag{12}$$

$$\int_0^{2\pi} Q_n\,dx = \int_0^{\infty}\int_{\psi}^{\infty}\int_0^{2\pi} R_n(x, \psi')\,dx\,d\psi'\,d\psi \qquad (n > 1). \tag{13}$$

From equation (12) we see immediately that the average value of the leading term Q_1 of the expansion for U^2 must be zero. As regards equation (13), each function R_n is defined in terms of the lower order coefficients $q_1, q_2, \ldots, q_{n-1}$ and hence is ultimately determined, on solving (10) successively for $q_1, q_2, \ldots, q_{n-1}$, by the lower order coefficients $Q_1, Q_2, \ldots, Q_{n-1}$. Equation (13) therefore implies a relation between each of the coefficients Q_n $(n > 1)$ of the expansion for U^2 and the earlier coefficients $Q_1, Q_2, \ldots, Q_{n-1}$. Unless the distribution of the external velocity round the outer edge of the boundary layer is such that the conditions represented by equations (12) and (13) are satisfied, the boundary layer motion with closed streamlines is not possible. The implication of these conditions for the inviscid flow will be discussed later. For the moment, it is assumed that the inviscid flow is such that they are satisfied.

In order to solve the equations for the coefficients q_n of the expansion for u^2, the functions that occur in these equations are expressed in Fourier series. Thus, we write

$$\left.\begin{aligned} q_n(x,\psi) &= \sum_{m=-\infty}^{\infty} q_{nm}(\psi)e^{imx}, \\ Q_n(x) &= \sum_{m=-\infty}^{\infty} Q_{nm}e^{imx}, \\ R_n(x,\psi) &= \sum_{m=-\infty}^{\infty} R_{nm}(\psi)e^{imx}, \end{aligned}\right\} \tag{14}$$

where, from the conditions (12) and (13) just derived,

$$Q_{10} = 0,$$
$$Q_{n0} = \int_0^\infty \int_\psi^\infty R_{n0}(\psi')\, d\psi' d\psi \qquad (n > 1).$$

The equations and boundary conditions for the coefficients q_n then reduce to the equations

$$\left.\begin{aligned} imq_{1m} &= imQ_{1m} + \frac{\partial^2 q_{1m}}{\partial\psi^2}, \\ imq_{nm} &= imQ_{nm} + \frac{\partial^2 q_{nm}}{\partial\psi^2} + R_{nm} \qquad (n > 1), \end{aligned}\right\} \tag{15}$$

with the boundary conditions for all n,

$$q_{nm}(0) = 0, \qquad \lim_{\psi\to\infty} q_{nm}(\psi) = Q_{nm}. \tag{16}$$

These equations define q_{nm} in terms of Q_{nm} and R_{nm}. On solving them and substituting in (14) we get

$$\left.\begin{aligned} q_1 &= \sum_{m=-\infty}^{\infty} Q_{1m}(1-e^{-\sqrt{(im)}\psi})e^{imx}, \\ q_n &= Q_{n0} - \int_\psi^\infty \int_\psi^\infty R_{n0}(\psi')\, d\psi' d\psi + \\ &\quad + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left\{ Q_{nm}(1-e^{-\sqrt{(im)}\psi}) + \frac{r_{nm}}{2\sqrt{(im)}} \right\} e^{imx} \qquad (n > 1), \end{aligned}\right\} \tag{17}$$

where

$$r_{nm} = e^{\sqrt{(im)}\psi} \int_\psi^\infty R_{nm}\, e^{-\sqrt{(im)}\psi}\, d\psi + \\ + e^{-\sqrt{(im)}\psi} \left\{ \int_0^\psi R_{nm}\, e^{\sqrt{(im)}\psi}\, d\psi - \int_0^\infty R_{nm}\, e^{-\sqrt{(im)}\psi}\, d\psi \right\}.$$

As may be recalled, Q_{nm} can be calculated directly from the external speed $U(x)$ (see (4) and (14)). Also $R_{nm}(\psi)$ can be calculated in terms of the coefficients $q_1, q_2, \ldots, q_{n-1}$ of lower order in n (see (8) and (14)). Equations (17), therefore, may be used to calculate in turn the successive coefficients q_n of the expansion for u^2.

When the coefficients q_n have been calculated in this way, the velocity component u may be determined. Also, the conditions (13) may be expressed directly in terms of the coefficients Q_{nm}. These conditions are now seen to be sufficient as well as necessary to allow q_n to be calculated. Subject to all the series used being convergent, the above calculation thus leads to a formal solution for the motion in the boundary layer and enables the necessary and sufficient conditions for the existence of the solution to be formulated in terms of the external velocity.

It remains to see how these results fit in with the expected behaviour of the inviscid motion. When an inviscid motion is bounded by surfaces whose position is known, there is in general one disposable constant of the inviscid motion for each closed surface. The conditions on Q_{nm} are clearly equivalent to a condition for such a disposable constant. Thus on the one hand the inviscid motion can generally be chosen so that the boundary layer motion of the type considered is possible. On the other hand, the existence of the closed boundary layer being assumed, the conditions on Q_{nm} provide just that extra information needed to determine the inviscid motion.

A formal solution having been obtained for a class of boundary layers whose streamlines are closed, it is of some interest to see whether the solution shows any features additional to those normally observed for boundary layers whose streamlines are open. Two such features seem worth mentioning. The first is the property that the speed at the outer edge of the boundary layer is related to the speed of the boundary. In assuming that the fluid just outside the boundary layer moves continuously round the boundary, it was already implicit that the circulatory movement of the boundary could impart circulation to the inviscid motion. Some such relation was, therefore, to be expected. The second feature concerns the velocity profile. Instead of tending monotonically to its limiting value in the mainstream, the magnitude of the velocity may oscillate infinitely often about this value. The underlying reason for this oscillatory behaviour is clear. If the velocity gradient $u\,\partial u/\partial\psi$ tended to zero monotonically, then on streamlines sufficiently far from the wall viscous action would always tend to accentuate the difference between the speed in the boundary layer and the speed just outside it. As a consequence, the speed on such streamlines would not return cyclicly to its initial value. Mathematically the oscillatory behaviour results from the periodic dependence of u on arc-distance, and bears an analogy to the oscillatory variation of amplitude with depth in the well-known 'skin effect' for a rapidly alternating electric current.

It may be noted in passing that the perturbation method used above extends to closed boundary layers in which the fluid is compressible. In this case the working assumption is that the motion in the boundary layer region differs only slightly from a motion with uniform speed and uniform temperature. In order for the closed motion to be possible, certain conditions must now be satisfied by the density and temperature just outside the boundary layer as well as by the speed there; the energy equation now giving rise to a set of conditions analogous to those obtained above from the momentum equation. Both the speed and the temperature may now oscillate about their values in the mainstream as the outer edge of the boundary layer is approached.

3. The condition on the external velocity distribution

As has been remarked, the condition on the external velocity distribution imposed by the closure of the boundary layer provides information needed to determine the inviscid motion. There seems little prospect of deriving the exact condition in a simple form. Relatively simple approximate forms can, however, be derived, and this we proceed to do.

The condition on the leading term of the expansion for the external velocity is given by (12). Correct to order γ this condition is equivalent to

$$\int_0^{2\pi} U\,dx = 2\pi. \tag{18}$$

The circulation round the boundary of the inviscid motion is thus specified directly. If the inviscid motion is confined by a single closed boundary at which this condition holds then the vorticity of the inviscid motion may also be specified directly. For, the streamlines being closed, the vorticity ω is uniform (Batchelor 1956a). Whence by Stokes's theorem,

$$\omega = \frac{2\pi L u_0}{A} + O(\gamma^2), \tag{19}$$

where A is the area occupied by the fluid. A similar result holds if the inviscid motion is confined by two closed boundaries and γ is a suitable expansion parameter for each simultaneously.

The conditions on the higher order terms of the expansion for the external velocity are defined by (13) together with the solution, now known, for the motion in the boundary layer. On substituting for R_2 and R_3 and using the governing equations for q_1 and q_2, it may be shown that

$$\int_0^{2\pi} (U^3 - U)\,dx = \tfrac{1}{4}\gamma^3 \int_0^{\infty}\int_{\psi}^{\infty}\int_0^{2\pi} \{Q_1(x) - q_1(x_1\psi')\}\,dx\,d\psi'\,d\psi + O(\gamma^4). \tag{20}$$

Then, when q_1 is replaced by the explicit expression obtained for it, this relation becomes

$$\int_0^{2\pi} (U^3 - U)\,dx = -\pi\gamma^3\mathscr{R}\left[\sum_{l=1}^{\infty} Q_{1l}\left\{\sum_{m=l}^{\infty} \frac{2(l-m)}{(\sqrt{l}+i\sqrt{m})^2} Q_{1m-l}\,\overline{Q}_{1m} + \sum_{m=1}^{\infty} \frac{l+m}{(\sqrt{l}+\sqrt{m})^2}\,\overline{Q}_{1m+l}\,Q_{1m}\right\}\right] + O(\gamma^4), \tag{21}$$

where the overbar denotes a complex conjugate. The condition that the boundary layer should close is thus expressed correct to order γ^3 as an explicit condition on the external speed.

Note that to order γ^2,

$$\int_0^{2\pi} (U-1)\,dx = -\frac{3}{2}\int_0^{2\pi} (U-1)^2\,dx < 0.$$

The circulation imparted to the inviscid flow is thus less than the circulation of the motion at the boundary. To order γ the circulation imparted is the same as that at the boundary, as may be seen directly from (18).

In several closed boundary layers to which these results apply, there is only one component in the Fourier series for $Q_1(x)$. Condition (21) then reduces to

$$\int_0^{2\pi} (U^3 - U)\, dx = O(\gamma^4). \tag{22}$$

4. Examples

To illustrate the way in which the additional boundary conditions just obtained may be used to determine the inviscid motion, two examples are considered.

The first concerns the motion of fluid enclosed within a fixed, nearly circular elliptic cylinder which is lined by a moving band. The band moves transversely to the generators and is assumed to maintain the fluid in a steady rotary motion in which the action of viscous forces is negligible save in a closed boundary layer on the band.

If the elliptic boundary were exactly circular, the fluid would rotate as a rigid body and at the boundary we would have $u = U = 1$. In the case of a nearly circular elliptic boundary, therefore, the velocity in the boundary layer may reasonably be expanded in powers of the small eccentricity e in the same way as in the general case the velocity was expanded in powers of γ.

Since the streamlines of the inviscid motion are closed, the vorticity of the inviscid motion is uniform (Batchelor 1956a). For a given value ω of this vorticity, the inviscid motion is determined by the condition of zero mass flux normal to the cylinder. It is supposed that the cylinder section has semi-axes a, b and is defined in Cartesian coordinates ξ, η by the equation

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1.$$

The stream function of the inviscid motion may then be written

$$\psi^* = -\frac{\omega a^2 b^2}{2(a^2+b^2)}\left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2}\right) + \text{constant}, \tag{23}$$

whence the speed of the inviscid motion at the boundary is

$$u_0\, U = \frac{\omega a^2 b^2}{a^2+b^2}\left(\frac{\xi^2}{a^4} + \frac{\eta^2}{b^4}\right)^{1/2}. \tag{24}$$

To determine the magnitude of the vorticity we use the further boundary condition imposed by the requirement that the boundary layer should close. As a preliminary we note that to order e,

$$U^2 = 1 + (Q_{10} + \tfrac{1}{2}\cos 2x)e.$$

Moreover, from the condition (12) on the leading term $Q_1(x)$, Q_{10} must be zero. The Fourier expansion of $Q_1(x)$ in this example thus contains only one term. The appropriate condition for the velocity at the outer edge of the boundary layer is, therefore, that given by (22). On substituting for U from (24) and integrating round the boundary, it follows immediately that correct to order e^3,

$$\omega^2 = 4u_0^2 \frac{(a^2+b^2)^3}{a^2b^2\{2(a^2+b^2)^2+(a^2-b^2)^2\}}.$$

Whence, after some reduction

$$\omega = \frac{2u_0}{\sqrt{(ab)}}\{1+O(e^4)\}. \tag{25}$$

This relation determines the vorticity and thence the stream function, of the inviscid motion correct to order e^3. To this order, the vorticity is apparently inversely proportional to the square root of the area occupied by the fluid.

Our second example concerns the steady streaming of unbounded fluid past a rotating cylinder. A series of photographs which illustrate this motion for various values of the peripheral speed u_0 relative to the mainstream speed U_∞ has been published by Prandtl & Tietjens (1936, plate 7). When u_0/U_∞ is small the boundary layer at the cylinder separates and a wake forms. As u_0/U_∞ increases, the size of the wake diminishes and ultimately, for u_0/U_∞ greater than 4, the boundary layer appears to be wholly attached to the cylinder. We assume, then, that for sufficiently large values of u_0/U_∞ the boundary layer remains attached to the cylinder.

The inviscid motion is then known apart from the value of its circulation Γ about the cylinder. At the cylinder, the speed of the inviscid motion is given by

$$u_0 U = \frac{\Gamma}{2\pi a} + 2U_\infty \sin x, \tag{26}$$

where a denotes the cylinder radius and x the inclination of the outward normal to the cylinder to the velocity of the mainstream.

If the fluid were at rest at infinite distance from the cylinder ($U_\infty = 0$), there would be no boundary layer at the cylinder and Γ would be equal to $2\pi a$. For values of U_∞ small compared to u_0, we may therefore expect that U^2 can be expanded in powers of U_∞/u_0 in the same way as it was formerly expanded in powers of γ.

To determine Γ we again use the additional boundary condition imposed by the closure of the boundary layer at the cylinder. As in the previous example, the appropriate condition is that given by (22). On substituting for U from (26) we get

$$\Gamma = 2\pi a u_0 \left\{1 - 3\frac{U_\infty^2}{u_0^2} + O\left(\frac{U_\infty^4}{u_0^4}\right)\right\}. \tag{27}$$

The lift coefficient for the cylinder, $C_L = L/\rho a U_\infty^2$, L being the lifting force and ρ the fluid density, is therefore given by

$$C_L = \frac{\Gamma}{aU_\infty} = \frac{2\pi u_0}{U_\infty}\left\{1 - 3\frac{U_\infty^2}{u_0^2} + O\left(\frac{U_\infty^4}{u_0^4}\right)\right\}. \tag{28}$$

5. THE CLOSED BOUNDARY LAYER WITH CONSTANT PRESSURE

So far the speed of motion round the outer edge of the boundary layer has been supposed to be non-uniform, the speed of motion at the boundary being uniform. It is now supposed that the speed $u_0 U$ of motion round the outer edge of the closed boundary layer is uniform, the speed $u_0 f(x)$ of motion at the boundary being non-uniform. Boundary layers of this kind are of interest because of the simplicity of the condition they impose on the external flow.

Since the pressure gradient of the external flow vanishes, the equation of motion in the boundary layer may be written

$$2\frac{\partial u}{\partial x} = \frac{\partial^2 u^2}{\partial \psi^2}. \tag{29}$$

On integrating round a closed streamline we get that

$$\frac{\partial^2}{\partial \psi^2}\int_0^{2\pi} u^2\, dx = 0. \tag{30}$$

Whence, on further integration,

$$U^2 = \frac{1}{2\pi}\int_0^{2\pi} f^2\, dx. \tag{31}$$

This is the condition that must be satisfied by the external speed if the boundary layer is closed and the pressure gradient is zero. Stated simply, the condition is that the speed of the external flow must be equal to the root-mean-square speed of the closed bounding surface.

It is rather interesting that this result is unimpaired if the boundary layer equation is replaced by the approximate equation

$$\frac{\partial u^2}{\partial x} = U\frac{\partial^2 u^2}{\partial \psi^2}. \tag{32}$$

Inasmuch as U is a (root-mean-square) average round each streamline of the quantity u which it replaces, this equation may well yield a solution more accurate than would normally be expected from a linearized form of the boundary layer equations.

As an example in which condition (31) may be used, we may mention the steady two-dimensional motion of fluid inside an infinite circular cylinder which is rotating about its axis, a fixed sheath shielding a segment of the boundary. The inviscid motion is then a rigid-body rotation. Thus the boundary layer on the sheath and exposed cylinder surface is closed and subject to zero pressure gradient. Hence from (31) the angular velocity of the rigid-body rotation Ω is related to the angular velocity of the cylinder Ω_0 by

$$\Omega = \sqrt{\left(\frac{\alpha}{2\pi}\right)}\,\Omega_0, \tag{33}$$

where α is the angle subtended at the axis by the unshielded segment of the cylinder surface.

The author first obtained the result (31) after seeing an early draft of a paper on closed flows by Dr Batchelor, and it has been reported in the published version of that paper (Batchelor 1956a). This relation and a weaker form of (21) have been independently obtained by Feynman & Lagerstrom, who reported their conclusions in a paper presented at the 9th International Congress for Applied Mechanics, Brussels, in September, 1956.

This example is also the one mentioned in the Introduction as having been considered by Squire. Squire used a Rayleigh-type approximation in which the boundary layer equation was, in effect, replaced by

$$u_0 U \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial y^2},$$

y being distance normal to the boundary. This equation yields the linear relation

$$\Omega = \frac{\alpha}{2\pi} \Omega_0, \tag{34}$$

which is, of course, correct when $\alpha = 0$ or 2π but elsewhere gives a value for Ω which is too small. The greatest error in Ω occurs when $\alpha = \frac{1}{2}\pi$, the result then being too small by a factor $\frac{1}{2}$.

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